Phase ordering in chaotic map lattices with conserved dynamics

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Dynamical scaling in a two-dimensional lattice model of chaotic maps, in contact with a thermal bath at temperature T, is numerically studied. The model here proposed is equivalent to a conserved Ising model with couplings that fluctuate over the same time scale as spin moves. When coupling fluctuations and thermal fluctuations are both important, this model does not belong to the class of universality of a Langevin equation known as model B; the scaling exponents are continuously varying with T and depend on the map used. The universal behavior of model B is recovered when thermal fluctuations are dominant. [S1063-651X(99)50711-3]

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The kinetics of phase separation as a system is quenched below its ordering temperature is the subject of much current research [1]. In the dynamical scaling regime the ordering process is characterized by a single time dependent length $R(t) \sim t^z$ representing the average domain size [2]; the growth exponent z usually does not depend on the dimensionality of the system nor on the final temperature of the quenching. In most models with nonconserved order parameter $z = 1/2$, while in scalar models with conserved parameters $z = 1/3$ [2]. A new exponent θ has been recently introduced by Derrida *et al.* [3] with the hope to better characterize the process of phase separation: it is related to the persistence probability $p(t)$ defined as the probability that the local order parameter at a given point has never changed since the initial time; typically one finds $p(t) \sim t^{-\theta}$ [4].

In a recent paper phase separation mechanisms have been investigated in the framework of coupled chaotic lattices [5], reaction-diffusion systems in which chaotic maps are coupled diffusively [6]. In the strong coupling regime these models exhibit nontrivial collective behavior, i.e., long-range order emerging out of local chaos. In [5] the one-body probability distribution functions (PDFS) of local (continuous) variables had two disjoint supports; this allowed the unambiguous definition of Ising spin variables. Starting from uncorrelated initial conditions, for large coupling values, complete phase ordering was observed; due to the deterministic nature of the system, the corresponding Ising spin system was at zero temperature. Both the domain size $R(t)$ and the persistence $p(t)$ showed scaling behavior and the exponents z and θ were found to vary continuously with parameters, at odds with traditional models. *Normal* phase ordering behavior, corresponding to the time dependent Ginzburg-Landau equation, was recovered in a continuous space limit of the model [7]. These findings suggest that the phase ordering properties of multiphase coupled chaotic lattices are different from those of most models studied traditionally. It is interesting, therefore, to investigate the nonuniversality of scaling exponents in *other* systems of chaotic maps.

In this work, we numerically study the dynamical scaling in a lattice model of chaotic maps such that the corresponding Ising spin model conserves the order parameter. By assuming the system to be in contact with a thermal bath, we also study the competition between the autonomous fluctuations of couplings and thermal fluctuations.

On each site i of a two dimensional square lattice, consider a dynamical system described by the variables x_i and y_i, which evolve according to the following map:

$$x_i(t+1) = \mathrm{erf}\left(\frac{\mu_i(t)}{\sigma_i(t)}\right) - \frac{1}{2}\left[\mathrm{erf}\left(\frac{\tau + \mu_i(t)}{\sigma_i(t)}\right) - \mathrm{erf}\left(\frac{\tau - \mu_i(t)}{\sigma_i(t)}\right)\right],$$
$$(1)$$

$$y_i(t+1) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{\tau + \mu_i(t)}{\sigma_i(t)}\right) + \mathrm{erf}\left(\frac{\tau - \mu_i(t)}{\sigma_i(t)}\right)\right], \quad (2)$$

where $\mu_i(t) = KJ_0 x_i(t)$, and $\sigma_i(t) = \sqrt{K[y_i(t) - J_0^2 x_i^2(t)]}$; K, J_0, and τ are free parameters. This map has been studied in [10] in the frame of chaotic neural networks; for a suitable choice of the parameters, it has two symmetric chaotic attractors, one with $x > 0$ and the other with $x < 0$. Hence, an Ising spin $\sigma_i(t) = \mathrm{sgn}[x_i(t)]$ can be associated with each dynamical system. At each discrete timestep t we assume the following dynamics, consisting of two stages: first, all the maps are iterated. Second, all pairs of nearest neighboring maps are sequentially swapped with the following exchange probability [11]: $P_{swap} = 1/(1 + \exp\{\beta \Delta E_t\})$, where β is the inverse temperature, ΔE_t is the change in energy that would occur if the maps were exchanged, and the configurational energy is defined as $E_t = -\Sigma_{\langle ij \rangle} x_i(t) x_j(t)$, the sum being over all the nearest neighboring pairs. It is clear that this dynamic conserves the spin magnetization $\Sigma \sigma_i$; by writing $E_t = -\Sigma_{\langle ij \rangle} J_{ij}(t) \sigma_i(t) \sigma_j(t)$, with $J_{ij}(t) = |x_i(t) x_j(t)|$, we observe that this model is equivalent to a conserved Ising model with couplings that fluctuate over the same time scale as spin moves. To study the phase ordering process, uncorrelated initial conditions were generated as follows: one half of the sites were chosen at random and the corresponding values of x and y were assigned according to the invariant distribution of the chaotic attractor with $x > 0$, while the other sites were similarly assigned values with $x < 0$. Large lattices (up to 1000×1000 sites) with periodic boundary conditions were used; the persistence $p(t)$ and the domain size $R(t)$ were measured. $R(t)$ was estimated by the condition $C[R(t), t] = 1/2$, where $C(\mathbf{r}, t) = \langle \sigma_{i+\mathbf{r}}(t) \sigma_i(t) \rangle$ is the two point correlation function. The persistence $p(t)$ was measured as the proportion of sites that has not changed σ from
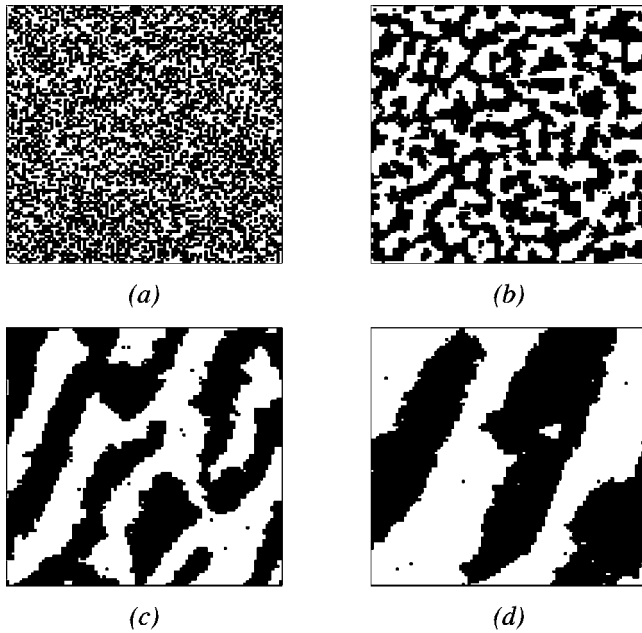
FIG. 1. Snapshots of the ordering system. Black (white) pixels correspond to $\sigma = 1(-1)$. Lattice of 100×100 sites, $\beta = 10$, and iterations times $t = 0$ (a), $t = 200$ (b), $t = 25\,000$ (c), and $t = 1\,000\,000$ (d).

FIG. 2. Time evolution of the domain size $R(t)$ (a) and persistence $p(t)$ (b) at zero temperature. Solid lines are best linear fits.

the initial time. Both $R(t)$ and $p(t)$ were averaged over many different samples of initial conditions.

Let us now discuss our results. In Fig. 1 typical configurations of the ordering system, at various times, are shown: the morphology of growing domains is similar to those in the conserved Ising model [9]. At zero temperature ($\beta = \infty$) the system coarsens and exhibits scaling behavior. Choosing $K = 10$, $\tau = 5$, and $J_0 = 0.9$, we measure $z = 0.11$ and $\theta = 0.59$. In Fig. 2(a) the time evolution of $R(t)$ is shown, while in Fig. 2(b) the time evolution of the persistence $p(t)$ is shown. We remark that the conserved Ising model with spin-exchange dynamics (which is recovered by suppressing couplings fluctuations in the present model) does not coarsen at zero temperature [9]. In the case of finite temperature, we find that the growth exponent varies continuously with β. In Fig. 3 we plot our measures of z for some values of the temperature $T = 1/\beta$. We observe that z increases monotonically with T and reaches the value $1/3$ at $T \approx 0.125$; it remains constant until $T = 0.32$. Therefore, in the range $T \in [0.125, 0.32]$, thermal fluctuations dominate over coupling fluctuations and the system belongs to the class of universality of a Langevin equation known as model B [8], which describes the standard conserved Ising model (when bulk diffusion dominate over surface diffusion [9]). At high temperatures ($T > 0.375$) the system does not order and $R(t)$ tends to a constant after a little transient ($z = 0$). For temperatures in the range $[0.32, 0.375]$, our numerical plots of $R(t)$ do not show a neat scaling behavior (filled area in Fig. 3): probably, in the thermodynamic limit a critical value β_c exists, in this range, separating the $z = 1/3$ behavior from the $z = 0$ one. Further numerical simulations are needed to study the phase ordering properties of this system close to β_c. We note that the order of magnitude of β_c estimated here is consistent with the critical coupling of the two-dimensional Ising model ($\beta_c = 0.44$) if one approximates the average
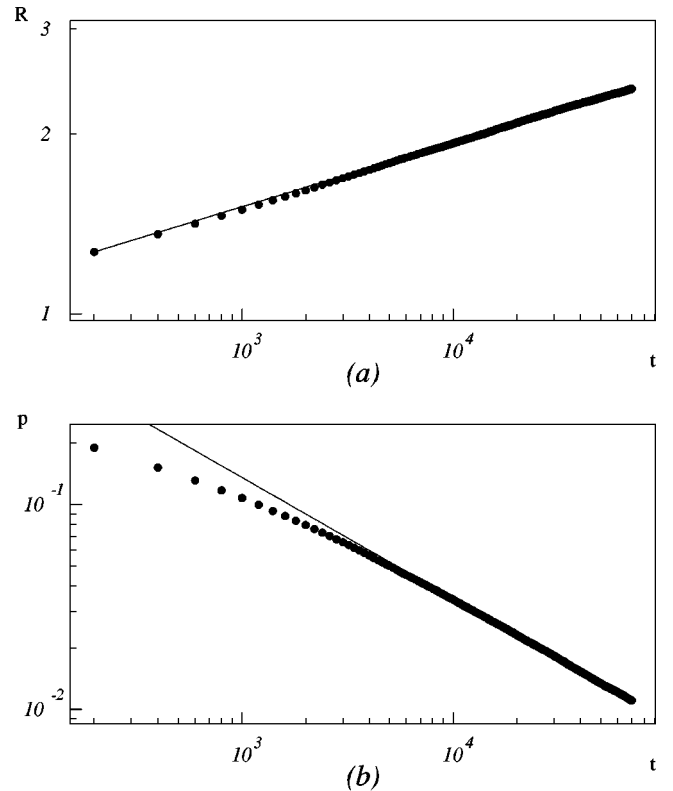
value of couplings as follows: $\langle J_{ij} \rangle \approx \langle |x_i(t)| \rangle \langle |x_j(t)| \rangle = 0.20$, the average of x's being calculated over the invariant distribution of the chaotic attractor.

It is worth stressing that simulations of the spin-exchange Ising model, at low temperatures, do not provide $z = 1/3$ but a value in the range $0.17–0.25$, because of corrections due to excess transport in interfaces [9]. We cannot exclude that these effects, peculiar to spin-exchange dynamics, still play a role, at low temperature, in the present model. However, we simulated, at low temperatures, the spin-exchange Ising model and, fitting the domain size with the power law form, we measured growth exponents differing from those measured in our model (using the same ratio T/T_c in the two models).

Now we consider the scaling function. In the scaling regime the correlation function obeys the following scaling form: $C(r,t) = f[r/R(t)]$. We remark that recently the universality of the scaling function f, with respect to details of the system, has been questioned [12]. In Fig. 4 the scaling collapse of the correlation function is depicted for three values of the temperature. At small r/R the scaling function has a linear behavior (Porod's law) $f \approx 1 - \alpha r/R$; we measure $\alpha \approx 0.5$, independently of the temperature. On the other hand, at larger distances f shows some dependence on the temperature (see Fig. 4): at the moment, we have no argument to explain this dependence. Concerning the persistence exponent, at finite temperature we measure the block persistence $p_{block}(t)$, which, for a sufficiently large block, is much less affected by thermal corrections than $p(t)$ [13]. We find that also θ depends on the temperature; for example at $\beta = 20$, using 8×8 blocks, we measure $\theta = 1.05$.
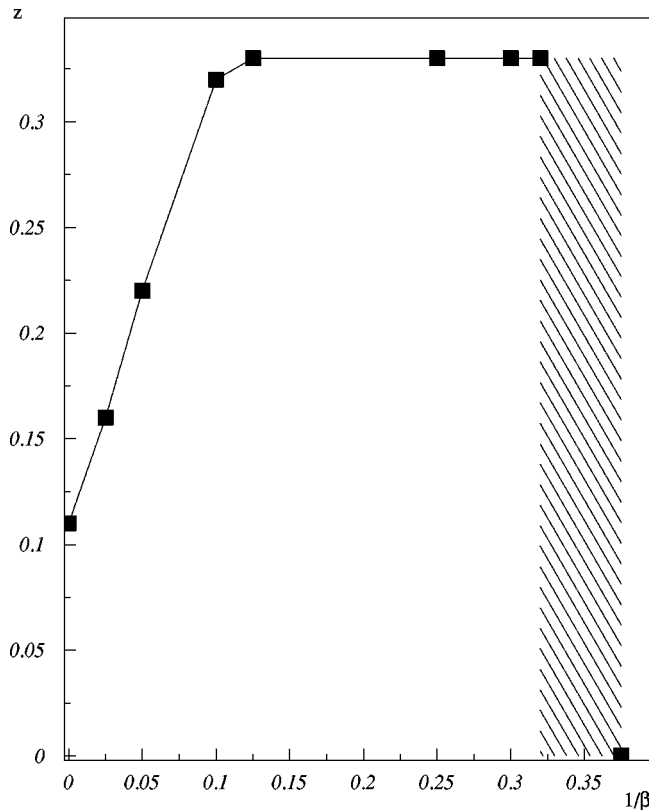
We also simulated systems with different maps. We do

FIG. 3. Estimated growth exponent z vs the temperature $T = 1/\beta$.

FIG. 4. Scaling collapse of the correlation function for $\beta = \infty$ (circles), $\beta = 20$ (squares), and $\beta = 10$ (triangles). Correlations at eight times, equally spaced in the interval $[10^4, 4 \times 10^4]$, are shown.

not report details here and only quote the results. We find that, at fixed temperature, the scaling exponents depend on the map. Changing the parameters in Eqs. (1) and (2) and fixing $K = 10$, $\tau = 5$, $J_0 = 0.95$, we measure, at zero temperature, $z = 0.22$ and $\theta = 1.57$; using the one-dimensional map of Ref. [5] [Eq. (2)], at zero temperature, we find $z = 0.07$ and $\theta = 0.09$. In both of these cases, however, we verify that the z exponent grows continuously with temperature and, when thermal fluctuations are dominant, z is close to 1/3. The scaling function f is found to be independent of the map at zero temperature.

In summary, we have numerically studied the dynamical scaling in a lattice model of chaotic maps which is equivalent to a conserved Ising model with fluctuating couplings, in contact with a thermal bath at temperature T. When coupling fluctuations and thermal fluctuations are both important, this model does not belong to the class of universality of model B: the scaling exponents are continuously varying with T and depend on the map used. The universal behavior of model B is recovered when thermal fluctuations are dominant. Our
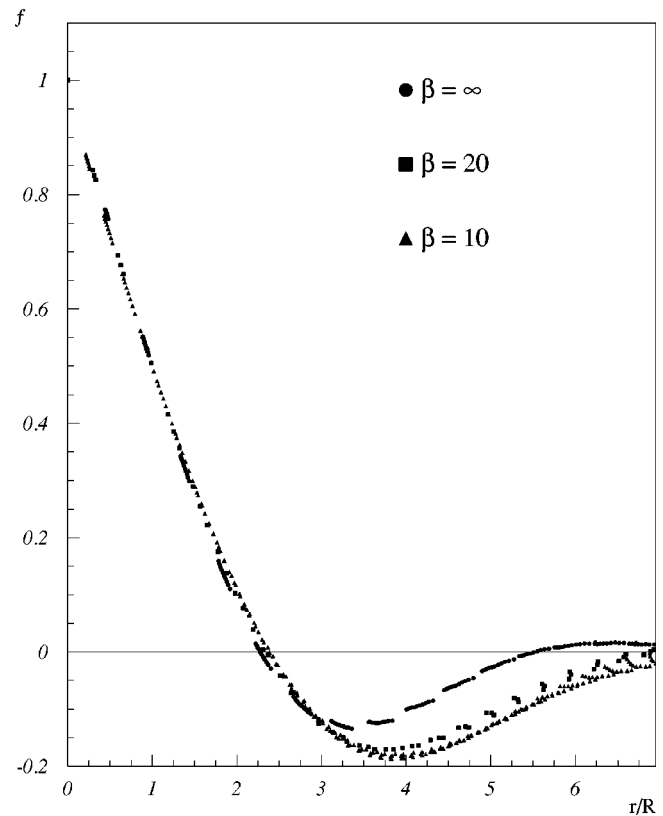
measures of the scaling function show some dependence on the temperature. A similar variation of the scaling exponents with parameters was found in [5], where the Ising models associated with the coupled map lattices were not conserving the order-parameter and at zero temperature: here we have confirmed this behavior in another class of chaotic map lattices, whose corresponding Ising models are conserving the order parameter and in presence of thermal fluctuations.

Finally, we observe that, in the long term, local chaos may be seen as bounded *effective* noise. It follows that a similar behavior might be found in the dynamical scaling of conserved coarsening systems in the presence of multiplicative noise [14], i.e., external fluctuations of a control parameter in the system. Further investigation, especially at the analytic level, is needed to clarify the origin of the nonuniversality described here.

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